

A Stationary Action Principle for the water sheet

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ABSTRACT. – We give a Stationary Action Principle formulation both for the dynamics and for the stationary state of a thin sheet of a non viscous fluid. We then derive the associated equations. © Elsevier, Paris.

1. Introduction

A water sheet is a thin layer of water moving in the air under the action of surface tension. The first experiments were due to Savart (1833). The equations describing the stationary state in the axisymmetrical case have been derived by Bousinesq (1869) and solved by Taylor (1959). The dynamical equations (also for non-Newtonian fluids) have been obtained by Entov (1987) in an intrinsic formalism (see also Yarin, 1993). The current interest in the problem is due to a large variety of phenomena such as nebulization, splashing of drops etc. (see Yarin, 1993 and references therein).

In this short note we formulate a simple Stationary Action Principle which allows us to derive the time evolution equations for any given parametric representation of the sheet. The interest of such a formulation lies in the clarity and simplicity of the approach and also because it suggests in a very natural way a particle method for approximate solutions.

2. The Stationary Action Principle

We represent the sheet at time t as

$$(2.1) \quad \{\mathbf{x}(r, s, t) \in \mathbb{R}^3 : (r, s) \in \Lambda \subset \mathbb{R}^2\}$$

where $(\mathbf{x}(r, s, \cdot))_{(r,s) \in \Lambda}$, are the Lagrangian variables. In other words $\mathbf{x}(r, s, t)$ is the position, at time t , of the particle of the sheet whose parameter at time zero is (r, s) . We define $m(r, s)$ as the mass density with respect to the measure $dr ds$. That is

$$(2.2) \quad m(r, s) dr ds = \rho d\sigma,$$

where ρ is the surface density of mass and $d\sigma$ is the area element on the surface of the sheet.

The kinetic energy K is

$$(2.3) \quad K \equiv \frac{1}{2} \int_{\Lambda} dr ds m(r, s) |\dot{\mathbf{x}}(r, s, t)|^2.$$

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The potential energy V is given by twice the area of the sheet multiplied by the surface tension T , that is

$$(2.4) \quad V \equiv 2T \int_{\Lambda} dr ds |\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}|$$

where $|\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}| dr ds = d\sigma$ is the surface element.

The Lagrangian is given by $L = K - V$, and the action between times t_1 and t_2 is $\mathcal{S} = \int_{t_1}^{t_2} dt L$, that is:

$$(2.5) \quad \mathcal{S} = \int_{t_1}^{t_2} dt \int_{\Lambda} dr ds \left(\frac{1}{2} m(r, s) |\dot{\mathbf{x}}(r, s, t)|^2 - 2T |\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}| \right).$$

Taking the formal variation of the action we obtain the Lagrange equations for the sheet:

$$(2.6) \quad m \ddot{\mathbf{x}} = 2T a \mathcal{H} \mathbf{n},$$

where the area element a , the normal \mathbf{n} , and the mean curvature \mathcal{H} , are given by

$$(2.7) \quad \begin{aligned} a &= |\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}| \\ \mathbf{n} &= \frac{\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}}{|\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}|} \\ \mathcal{H} &= \frac{|\partial_r \mathbf{x}|^2 \partial_s^2 \mathbf{x} \cdot \mathbf{n} - 2 (\partial_r \mathbf{x} \cdot \partial_s \mathbf{x}) \partial_{rs}^2 \mathbf{x} \cdot \mathbf{n} + |\partial_s \mathbf{x}|^2 \partial_r^2 \mathbf{x} \cdot \mathbf{n}}{|\partial_r \mathbf{x} \wedge \partial_s \mathbf{x}|} \end{aligned}$$

Remarks. – (i) Notice that $\mathcal{H} \mathbf{n}$ is invariant for changes in the parameterization of the surface. For a this is not the case.

(ii) In the potential energy V the area of the surface is multiplied by $2T$ as follows by considering a thin layer of fluid in the limit of vanishing thickness, see Benedetto (1995); in this case there are two surfaces that, in the limit, coincide with that of the sheet.

(iii) The description given above works in the case in which the surface has no boundaries. In the general case it is necessary to give a model for the evolution of the boundary. This is not trivial and various phenomenological models have been proposed (see Taylor, 1959; Yarin, 1993).

(iv) A particle method devoted to numerical simulation of (2.6) may be obtained by discretizing the Lagrangian. In this way the particle system is interacting via the potential obtained by discretizing the area of the surface.

(v) In case where the surface is translational invariant along some direction, we can choose r as the abscissa on the invariant direction and use s to represent the intersection of the surface with an orthogonal plane. Therefore the problem is two-dimensional and the Lagrangian is

$$(2.8) \quad L = \int_{s_1}^{s_2} ds \left(\frac{1}{2} m(s) |\dot{\mathbf{x}}(s, t)|^2 - 2T |\partial_s \mathbf{x}| \right).$$

Notice that in this case the potential energy is proportional to the length of the curve.

The Lagrange equations are

$$(2.9) \quad m \ddot{\mathbf{x}} = 2T \partial_s \left(\frac{\partial_s \mathbf{x}}{|\partial_s \mathbf{x}|} \right).$$

3. The stationary case

We look for stationary flows that is situations in which the fluid describes a surface constant in time and the velocity field on the surface is constant. Also the surface mass density is constant.

In order to describe a stationary flow in a Lagrangian formalism, we choose the parameter s as a coordinate on the time trajectory. Then $m(r, s)$ does not depend on s . Furthermore we note that a fluid particle $\mathbf{x}(r, s, t)$ is moving on the stationary surface and it feels a force which is orthogonal to its path, because the force is orthogonal to the surface (see Eq. (2.6)). Therefore $|\dot{\mathbf{x}}(r, s, t)|$ is conserved and is a function of r alone; let us name this function $v(r)$. Therefore a stationary flow is given by the condition:

$$(3.1) \quad \mathbf{x}(r, s, t) = \mathbf{y}(r, s - v(r)t).$$

By substituting (3.1) in the equations of motion (2.6–7), we find

$$(3.2) \quad m v^2 \partial_s^2 \mathbf{y} = 2T a \mathcal{H} \mathbf{n},$$

where the area element a , the normal \mathbf{n} , and the mean curvature \mathcal{H} , are given by

$$(3.3) \quad \begin{aligned} a &= |\partial_r \mathbf{y} \wedge \partial_s \mathbf{y}| \\ \mathbf{n} &= \frac{\partial_r \mathbf{y} \wedge \partial_s \mathbf{y}}{|\partial_r \mathbf{y} \wedge \partial_s \mathbf{y}|} \\ \mathcal{H} &= \frac{|\partial_r \mathbf{y}|^2 \partial_s^2 \mathbf{y} \cdot \mathbf{n} - 2(\partial_r \mathbf{y} \cdot \partial_s \mathbf{y}) \partial_{rs}^2 \mathbf{y} \cdot \mathbf{n} + \partial_r^2 \mathbf{y} \cdot \mathbf{n}}{|\partial_r \mathbf{y} \wedge \partial_s \mathbf{y}|}. \end{aligned}$$

Moreover it is easy to see that (3.2) are the Euler-Lagrange equations for the functional \mathcal{A} :

$$(3.4) \quad \mathcal{A} = \int_{s_1}^{s_2} ds \int dr \left(\frac{1}{2} m(r) v^2(r) |\partial_s \mathbf{y}(r, s)|^2 - 2T |\partial_r \mathbf{y} \wedge \partial_s \mathbf{y}| \right).$$

An example: the Water Bell

A water bell is a bell shaped water sheet. It can be obtained by placing a disk shaped obstruction in the path of a cylindrical jet of water (see Savart, 1833; Taylor, 1959). In this case the surface is axially symmetric and we use cylindrical coordinates r, z, θ , where r is the distance from the symmetry-axis, z is the coordinate along the axis, and θ is the angle of the orthogonal plane to the axis. Then our unknowns are $r(s), z(s), \theta(s)$.

The Lagrangian is

$$(3.5) \quad L = \frac{1}{2} \mu ((\partial_s r)^2 + (\partial_s z)^2 + r^2 (\partial_s \theta)^2) - 2Tr \sqrt{(\partial_s r)^2 + (\partial_s z)^2};$$

where μ is constant.

The solution of this problem is an easy exercise of Lagrangian mechanics: we have in fact a three degree of freedom Lagrangian system and two variables, z and θ , do not appear in the Lagrangian. Therefore we have three conserved quantities: the energy

$$(3.6) \quad E = \frac{\partial L}{\partial(\partial_s r)} \partial_s r + \frac{\partial L}{\partial(\partial_s z)} \partial_s z + \frac{\partial L}{\partial(\partial_s \theta)} \partial_s \theta - L = \frac{1}{2} \mu ((\partial_s r)^2 + (\partial_s z)^2 + r^2 (\partial_s \theta)^2),$$

the impulse related to θ (the angular momentum), and the impulse related to z :

$$(3.7) \quad J = \frac{\partial L}{\partial(\partial_s \theta)} = \mu r^2 \partial_s \theta, \quad I = \frac{\partial L}{\partial(\partial_s z)} = \partial_s z \left(\mu - \frac{2Tr}{\sqrt{(\partial_s r)^2 + (\partial_s z)^2}} \right).$$

We are not interested in $r(s)$, $z(s)$, $\theta(s)$ but in $r(z)$. Denoting $r_0 = \frac{J}{\sqrt{2\mu E}}$, $\bar{r} = \frac{\sqrt{2\mu E}}{2T}$, from eqs. (3.6–7) we obtain:

$$(3.8) \quad I = \mu \partial_s z \frac{\sqrt{1 - \frac{r_0^2}{r^2} - \frac{r}{\bar{r}}}}{\sqrt{1 - \frac{r_0^2}{r^2}}}.$$

Using (3.8) we can eliminate $\partial_s z$ in the expression for the energy

$$(3.9) \quad E = \frac{1}{2} \mu (\partial_s z)^2 (1 + (\partial_z r)^2) + \frac{J^2}{2\mu r^2}.$$

obtaining:

$$(3.10) \quad 1 + (\partial_z r)^2 = \frac{2\mu E}{I^2} \left(\sqrt{1 - \frac{r_0^2}{r^2} - \frac{r}{\bar{r}}} \right)^2.$$

In the case $J = 0$ (and then $r_0 = 0$), equation (3.10) admits the general solution

$$(3.11) \quad r(z) = \bar{r} \pm \alpha \cosh\left(\frac{z}{\alpha}\right).$$

where $\alpha = \frac{I}{2T}$. These are the solutions found by Taylor (1959), where we refer the reader for a physical discussion.

The case $J \neq 0$ is a bit more complicated. In particular by varying J it is possible distinguish to different regimes. Here we consider the case $\bar{r} > 2r_0$, that seems to be the more relevant for the application (see Boussinesq, 1869; Taylor, 1959; Yarin, 1993). In this case the level sets of $\frac{2\mu E}{I^2}$ shows three different regimes separated by $r = r_- = \bar{r} \sqrt{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{r_0^2}{\bar{r}^2}}}$, and $r = r_+ = \bar{r} \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{r_0^2}{\bar{r}^2}}}$. The first, $r < r_-$, and the third, $r > r_+$ corresponds qualitatively to the two regimes (– and +) for the case $J = 0$.

In the intermediate case, $r_- < r < r_+$, the orbits are periodic; this corresponds, in the real space, to z -periodic surfaces; in particular there exists r_c such that for $r(z) \equiv r_c$ the solution is a stable cylinder.

For more details see Gasser (1994), where the case $J \neq 0$ has been extensively studied.

Remark. – The problem of deriving the equation of the water-sheet from the Euler equations for an incompressible non viscous fluid is interesting and unsolved. In Benedetto (1995) we discuss a possible way to approach this problem.

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